Last Time: Derivatives of Multivariable functions

Directional Derivative: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u})-f(\vec{a})}{h}$

unit vector in $\mathbb{R}^n$

$\vec{a} \in dom(f)$

n-variable function

Defn: the k^{th} partial derivative (or the partial derivative wrt x_k) of n-variable f is $\frac{\partial f}{\partial x_k} = D_{\vec{e}_k} f$ where

$$\vec{e}_k = \langle \underbrace{0, \ldots}_{\text{all } 0}, \underset{k^{th} \text{ position}}{1}, \underbrace{\ldots, 0}_{\text{all } 0} \rangle$$

NB: $\vec{e}_k$ is the increasing direction for x_k where $\mathbb{R}^n$ has coordinates $(x_1, x_2, \ldots, x_n)$

What's going on?: Two variables (x, y)
Given function $f(x, y)$ and $(a, b) \in dom(f)$

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = D_{\vec{e}_2} f(a, b) \qquad e_2 = \langle 0, 1 \rangle$$

y is second coordinate

$$= \lim_{h \to 0^+} \frac{f(\langle a,b \rangle + h\vec{e}_2) - f(\langle a,b \rangle)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(a + h \cdot 0,\ b + h \cdot 1) - f(a,b)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(a, b+h) - f(a,b)}{h}$$ ← first coordinate not changing (ie held constant)

Now let $g(y) := f(a, y)$ single variable function

rewrite $\left.\frac{df}{dy}\right|_{(a,b)} = \lim_{h \to 0^+} \frac{f(a, b+h) - f(a,b)}{h}$

$$= \lim_{h \to 0^+} \frac{g(b+h) - g(b)}{h} = g'(b)$$ calculus I derivative

By construction g treats x as a constant so g' is the derivative of f "pretending" x is constant

* This works similarly for every component

Ex: Take all partial derivatives of $f(x,y) = xy^2 - x^{3/2} + \sin(x-y)$

sol $\frac{df}{dx} = \frac{d}{dx}\left[xy^2 - x^{3/2} + \sin(x-y)\right]$

usual derivative so all usual rules apply

$$= \frac{d}{dx}\left[xy^2\right] - \frac{d}{dx}\left[x^{3/2}\right] + \frac{d}{dx}\left[\sin(x-y)\right]$$

$$= y^2 \frac{d}{dx}\left[x\right] - \frac{3}{2}x^{\frac{1}{2}} + \cos(x-y)\frac{d}{dx}\left[x-y\right]$$

$$= y^2 - \frac{3}{2}x^{\frac{1}{2}} + \cos(x-y)$$

$$\frac{df}{dy} = \frac{d}{dy}\left[xy^2 - x^{\frac{3}{2}} + \sin(x-y)\right]$$

$$= \frac{d}{dy}\left[xy^2\right] - \frac{d}{dy}\left[x^{\frac{3}{2}}\right] + \frac{d}{dy}\left[\sin(x-y)\right]$$

$$= x\frac{d}{dy}\left[y^2\right] - 0 + \cos(x-y)\frac{d}{dy}\left[x-y\right]$$

$$= x(2y) + \cos(x-y)(-1)$$

$$= 2xy - \cos(x-y) \quad \boxed{///}$$

<u>Ex</u>: Compute partial derivatives of
$f(x,y,z) = e^{x^2+y^2}\sin(xz)\cos(yz)$

<u>Sol</u>: $\frac{df}{dx} = \frac{d}{dx}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right]$

$$= \cos(yz)\frac{d}{dx}\left[e^{x^2+y^2}\sin(xz)\right]$$

$$= \cos(yz)\left(\frac{d}{dx}\left[e^{x^2+y^2}\right]\sin(xz) + e^{x^2+y^2}\frac{d}{dx}\left[\sin(xz)\right]\right)$$

$$= \cos(yz)\left(2xe^{x^2+y^2}\sin(xz) + e^{x^2+y^2}\cos(xz)z\right)$$

$$= \cos(yz)e^{x^2+y^2}\left(2x\sin(xz) + z\cos(xz)\right)$$

$$\frac{df}{dy} = \frac{d}{dy}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right]$$

$$= \sin(xz)\frac{d}{dy}\left[e^{x^2+y^2}\cos(yz)\right]$$

$$= \sin(xz)\left(\frac{d}{dy}\left[e^{x^2+y^2}\right]\cos(yz) + e^{x^2+y^2}\frac{d}{dy}\left[\cos(yz)\right]\right)$$

$$= \sin(xz)\left(2ye^{x^2+y^2}\cos(yz) + e^{x^2+y^2}\left(-\sin(yz)z\right)\right)$$

$$= e^{x^2+y^2}\sin(xz)\left(2y\cos(yz) - z\sin(yz)\right)$$

$$\frac{df}{dz} = \frac{d}{dz}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right]$$

$$= e^{x^2+y^2}\frac{d}{dz}\left[\sin(xz)\cos(yz)\right]$$

$$= e^{x^2+y^2}\left(\frac{d}{dz}\left[\sin(xz)\right]\cos(yz)+\sin(xz)\frac{d}{dz}\left[\cos(yz)\right]\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz)+\sin(xz)(-y\sin(yz))\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz)-y\sin(xz)\sin(yz)\right)$$

$\boxtimes$

<u>NB</u>: Everything w/ partial derivatives is working out mostly the same as calculus I, once we hold variables constant...

We can make second order derivatives in exactly the same way as we did in calculus I... now there's just more

derivative of derivative

$$\frac{d^2f}{(dx)^2},\ \frac{d^2f}{(dy)^2} \qquad \frac{d^2f}{dydx},\ \frac{d^2f}{dxdy}$$

"pure second order partials" "mixed second order partials"

Ex: Compute second order partial derivatives of $f(x,y) = xy^2 - x^{3/2} + \sin(x-y)$

sol: We computed earlier

$$\frac{df}{dx} = y^2 - \frac{3}{2}x^{\frac{1}{2}} + \cos(x-y) \quad \text{and}$$

$$\frac{df}{dy} = 2xy - \cos(x-y)$$

Now we compute

$$\frac{d^2f}{(dx)^2} = \frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{d}{dx}\left[y^2 - \frac{3}{2}x^{\frac{1}{2}} + \cos(x-y)\right] = \frac{-3}{4}x^{-\frac{1}{2}} - \sin(x-y)$$

$$\frac{d^2f}{(dy)^2} = \frac{d}{dy}\left[\frac{df}{dy}\right] = \frac{d}{dy}\left[2xy - \cos(x-y)\right]$$

$$= 2x - \sin(x-y)$$

Moreover, we have mixed partials

$$\frac{d^2f}{dy\,dx} = \frac{d}{dy}\left[\frac{df}{dx}\right]$$

$$= \frac{d}{dy}\left[y^2 - \frac{3}{2}x^{1/2} + \cos(x-y)\right]$$

$= 2y - 0 - \sin(x-y)(-1) = 2y + \sin(x-y)$

$$\frac{d^2 f}{dx\,dy} = \frac{d}{dx}\left[\frac{df}{dy}\right]$$

$$= \frac{d}{dx}\left[2xy - \cos(x-y)\right] = 2y - (-\sin(x-y) \cdot 1)$$

$= 2y + \sin(x-y)$ $\boxtimes$

NB: Up to this point, applying partial derivatives 'just works' in exactly the same way as calculus I...

Want: Understand mixed partial derivatives

(1) Why did this example have $\frac{d^2 f}{dx\,dy} = \frac{d^2 f}{dy\,dx}$?

(2) How can we guarantee (or tell in advance) if this happens for future functions?

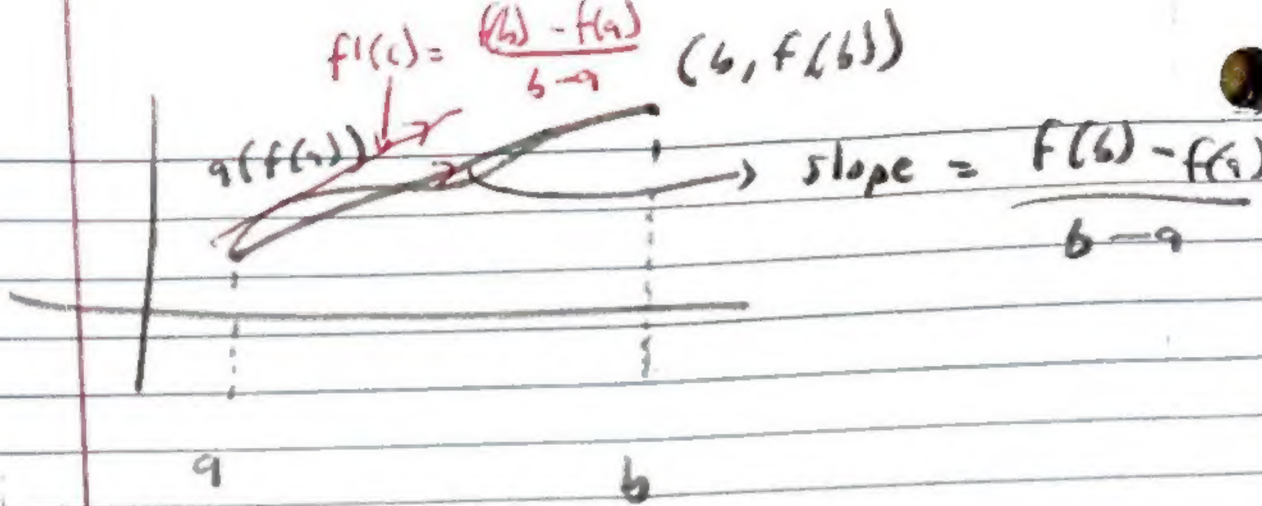
To answer these questions, we need to recall some Calculus I ...

$\rightarrow$ MVT

Prop (Mean Value Theorem):

Let $f(t)$ be a function which is diff on (a,b) and cts on $[a,b]$

Then there is $a < c < b$ such that
$f'(c)(b-a) = f(b) - f(a)$



Answer the mixed partials question uses MVT.

<u>Prop (Clairaut's Theorem)</u>: Let $f(x,y)$ have cts second order mixed partial derivatives on a disk containing (a,b). Then at (a,b) we have

$$\frac{d^2 f}{dx\,dy}\Big|_{(a,b)} = \frac{d^2 f}{dy\,dx}\Big|_{(a,b)}$$